

# Reduced quantum dynamics with initial system-environment correlations characterized by pure Markov states

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Any tripartite state which saturates the strong subadditivity relation for the quantum entropy is defined as the Markov state. A tripartite pure state describing an open system, its environment and their purifying system is a pure Markov state iff the bipartite marginal state of the purifying system and environment is a product state. It has been shown that as long as the purification of the input system-environment state is a pure Markov state the reduced dynamics of the open system can be described, on the support of initial system state, by a quantum channel for every joint unitary evolution of the system-environment composite even in the presence of initial correlations. Entanglement, discord and classical correlations of the initial system-environment states implied by the pure Markov states are analyzed and it has been shown that all these correlations are entirely specified by the entropy of environment. Some implications concerning perfect quantum error correction procedure and quantum Markovian dynamics are presented.

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## I. INTRODUCTION

Each real world quantum system forms a closed compound system with its surrounding environment and evolves together with it under joint unitary evolutions. Such systems are called open quantum systems (OQSs) [1, 2]. Each OQS interacts and gets correlated to some extent with its environment and evolves according to quantum rules individually, at least for a while, before completely losing its quantum coherence property [2]. On the other hand, the main goal in realizing many quantum information and quantum computation tasks is to maintain, as long as possible, the coherence properties of information carriers which are constantly interacting with ambient medium [3]. However, despite their importance in our understanding the quantum aspects of the nature around us and in the emerging fields of quantum technologies we still lack a complete understanding of evolutions of OQSs initially correlated with their environment.

Physically most appealing way to describe evolutions of an OQS is by the so-called completely positive (CP) maps [4, 5]. These are linear maps which transform every positive operators in their definition domain to positive operators and maintain this property in all tensorial extensions. Trace preserving CP maps are called quantum channels and they are interchangeably referred to also as CPTP maps: These map any quantum state (density operator; positive operator with unit trace) to another quantum state [6]. To emphasize one of the physical in-

tuitions behind such maps let us consider a given CPTP map acting on an OQS. By appending an auxiliary system, say the environment of OQS, the tensorial extension of such a map can be defined on any joint quantum state of the compound system, the OQS and auxiliary system. The result of extended action is certainly another admissible joint state and this is what complete positivity corresponds to in applications. Moreover, when the effects of appended system are then averaged out what remains is the action of the same CPTP map on the reduced state of the original OQS. This holds irrespective of the correlations the initial joint state may have and of the dimension of added system provided that the original state of OQS is the reduced state of the joint state.

In real world and in the laboratory applications however CP maps are reduced from the joint unitary evolutions and the essential problems arise in this context. In such a case the action of a joint unitary map on a joint state may not give the action of an even positive map, let alone CP map, on the reduced state of the OQS after discarding the environment [7] (see also [8]). Certainly, uncorrelated joint states, that is, product states are exceptions and starting with such a state has become a basic assumption in almost all approaches to the dynamics of OQSs.

In fact, there is a whole set of exceptions that provides a large family of initially correlated joint states, not recognized in the literature before the recent work [9]. The main goal of our study is to specify such a well defined special subset of correlated initial system-environment states that not only permits CP reduced dynamics for the observed OQS, but also makes it possible to characterise all classical and quantum correlations of its elements. The states that will be explored here are tripartite states that can be reconstructed from their marginal states via the actions of CP maps. Since aver-

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aging the effect of a subsystem is carried out by a partial trace, such a CP map locally reverses the action of partial trace map on the considered state. Evidently, a product joint state is such a state since tensoring by an additional state is a CPTP map on the other factor and therefore this family contains uncorrelated initial states as special cases.

Reconstruction positive maps were also known in the context of OQSs under the name of the assignment maps but, unfortunately, they were not explored sufficiently enough. A detailed study of reconstruction CP maps were made in another context; in characterising tripartite states that saturate the strong subadditivity (SSA) relation for the (von Neumann) quantum entropy [10–12] (and the references therein). In this context such a map is known as the Petz recovery map and related tripartite states are called Markov states. Very recently, by including an additional blind and dead reference system into discussion F. Buscemi has shown in Ref. [9] that the reduced dynamics of an OQS can be described by CP maps in the presence of initial correlations. Such a description is possible for tripartite states of the reference system, the OQS and the environment trio for which the quantum mutual information between the reference system and the environment, conditional on the system, is zero. (Positivity of this mutual information is better known as the SSA relation [3, 10].) It should be noted that the system-environment states which are marginals of tripartite input Markov states, conditional on system are not the only states for which the system evolution is given by a CPTP linear map.

In this study, we shall use a similar tripartite framework of Ref. [9] but for a detailed exposition of the problems and in order to be able to analyze the initial correlations explicitly, we shall restrict our consideration mainly to pure Markov states. A direct and detailed study of these states and of related quantum channels as well as qualitative and quantitative characterizations of all possible classical and quantum correlations of the initial system-environment states are, to the best of our knowledge, new contributions of this study to the present OQS literature.

## II. FRAMEWORK OF THIS STUDY AND SUMMARY

We shall denote the OQS and its environment, respectively, by  $Q$  and  $E$  and suppose that the bipartite system  $QE$  forms a closed system subject to time-dependent joint unitary operator  $U^{QE}$ . The purifying system of  $QE$  states will be represented by  $R$ . Each system is supposed to be endowed with a finite dimensional Hilbert space  $\mathcal{H}_X$  and with the space  $B(\mathcal{H}_X)$ ;  $X = R, Q, E$ , of bounded operators. If we denote the dimension of  $\mathcal{H}_X$  by  $d_X$ , then  $d_R$  is not smaller than  $d_Q d_E$ .

Even when  $Y$  represents a compound system, its quantum states will be denoted by density operators  $\rho^Y$ . The

tensorial extension to  $B(\mathcal{H}_Q \otimes \mathcal{H}_E) = B(\mathcal{H}_{QE})$  of a map  $\Lambda$  defined on  $B(\mathcal{H}_Q)$  will be denoted by  $\Lambda \otimes id_E, id_X$  being the identity map on  $B(\mathcal{H}_X)$ . If  $\Lambda$  is a positive map and if  $\Lambda \otimes id_E$  preserves the positivity of operators defined on  $B(\mathcal{H}_{QE})$  for all dimensions of  $\mathcal{H}_E$ ,  $\Lambda$  is a CP map. Partial traces and adjoint actions of unitary operators are the standard examples of CPTP maps.

Throughout this study the initial and final states will be denoted by indexed  $|\psi\rangle, \rho$  and  $|\phi\rangle, \sigma$ , respectively. Superscripts over them will indicate which system they belong to. They should be thought of also as indexed by an initial time  $\tau$  and final time  $\tau' \geq \tau$ . Accordingly,  $U^{QE}$  and related channels should be thought of as indexed by both times leading to state vectors and operators at the initial time to that of at the final time. At the beginning  $QE$  is supposed to be in the reduced state  $\rho^{QE} = Tr_R |\psi^{RQE}\rangle \langle \psi^{RQE}|$  of the initial tripartite pure state  $|\psi^{RQE}\rangle$ . Evidently the rank of  $\rho^R$ , that is, the number of nonzero eigenvalues of  $\rho^R = Tr_{QE} |\psi^{RQE}\rangle \langle \psi^{RQE}|$ , is always equals to the rank of  $\rho^{QE}$ :  $rank(\rho^{QE}) = rank(\rho^R)$ .

Defining the adjoint action of an operator  $V$  on  $\rho$  by  $ad_V(\rho) = V\rho V^\dagger$ , the output state is  $\sigma^{RQE} = (id_R \otimes ad_{U^{QE}})\rho^{RQE}$ . Hence  $R$  remains blind and dead during the evolution. Since the overall evolution is unitary, when the input state is pure  $\rho^{RQE} = |\psi^{RQE}\rangle \langle \psi^{RQE}|$  then so is the output  $\sigma^{RQE} = |\phi^{RQE}\rangle \langle \phi^{RQE}|$  where  $|\phi^{RQE}\rangle = (\mathbb{I}_R \otimes U^{QE})|\psi^{RQE}\rangle$  ( $\mathbb{I}_X$  stands for the unit operator, or the unit matrix of  $\mathcal{H}_X$ ). In any case, the reduced dynamics of  $Q$  is specified by tracing out the environment and the purifying system:

$$\sigma^Q = \mathcal{E}(\rho^Q) = Tr_{RE}(\sigma^{RQE}). \quad (1)$$

Now the important question is that, for what kind of initial correlations of  $\rho^{QE}$  is the map  $\mathcal{E}$  a linear CPTP map?

As a non-exhaustive answer to the above question in what follows we shall prove that as long as the input  $QE$  state is a reduced state of a pure Markov state the evolution map  $\mathcal{E}$  is a CPTP map for every joint unitary evolution of the  $QE$  composite in the presence of initial correlations implied by the pure Markov state. This is shown in Sec. IV where the explicit form of the channel and its Kraus operators as well as identification some special cases of channel are presented. In Sec. III the necessary entropy relations, Markov states, Petz map and pure Markov states are introduced. Canonical form of the pure Markov states and their characteristic traits are also exhibited in Sec. III. The correlations such as entanglement of formation, discord and classical correlations that the initial  $QE$  states may have are analyzed in Sec. V. There it is shown that the entropy of the environment entirely specifies all these correlations. Our main points concerning the CPTP evolutions and characterizations of initial correlations are summarized by two theorems. In the final section intimate connections of our results with perfect quantum error correction procedure and quantum Markovian dynamics are discussed.

### III. SSA RELATION, MARKOV STATES, PETZ MAP AND PURE MARKOV STATES

The von Neumann entropy of a state  $\rho^Y$  is defined by  $S(\rho^Y) = -\text{Tr} \rho^Y \log \rho^Y$ . This will be denoted simply by  $S(Y) = S(\rho^Y)$ . The quantum conditional entropy  $S(X|Y) = S(XY) - S(Y)$  and the quantum mutual information  $S(X;Y)$  defined by

$$S(X;Y) = S(X) + S(Y) - S(XY), \quad (2)$$

will be distinguished with special punctuation inside the parenthesis.  $S(X;Y)$  is zero iff  $\rho^{XY}$  is the product state  $\rho^{XY} = \rho^X \otimes \rho^Y$ , where  $\rho^X = \text{Tr}_Y \rho^{XY}$  and  $\rho^Y = \text{Tr}_X \rho^{XY}$  are the marginal states of  $\rho^{XY}$ .

Accordingly, the conditional mutual information  $S(R;E|Q)$ , conditioned on  $Q$ , for a state of a tripartite system  $RQE$  is defined as  $S(R;E|Q) = S(R|Q) + S(E|Q) - S(RE|Q)$ . In view of the definition of conditional entropy this takes the form

$$S(R;E|Q) = S(RQ) + S(QE) - S(RQE) - S(Q), \quad (3)$$

and the celebrated SSA relation which hosts several entropy relations can be expressed by  $S(R;E|Q) \geq 0$ .

A tripartite state is a Markov state conditional on  $Q$  iff it satisfies  $S(R;E|Q) = 0$ . A key property of Markov states we shall use is that:  $\rho^{RQE}$  is a Markov state iff there exists a CPTP map  $\mathcal{R} : Q \rightarrow QE$  such that

$$\rho^{RQE} = (id_R \otimes \mathcal{R})\rho^{RQ},$$

where  $\rho^{RQ} = \text{Tr}_E \rho^{RQE}$  (Eq. (11) of Ref. [10] see also [11]). On the support of  $\rho^Q = \text{Tr}_{RE} \rho^{RQE}$  the action of  $\mathcal{R}$  on any  $X \in B(\mathcal{H}_Q)$  is given by (Eq. (15) of Ref. [13])

$$\mathcal{R}(X) = ad_{(\rho^{QE})^{1/2}}[(ad_{(\rho^Q)^{-1/2}}X) \otimes \mathbb{I}_E]. \quad (4)$$

Note that for  $X = \rho^Q$  we have  $\mathcal{R}(\rho^Q) = \rho^{QE}$ . That is,  $\mathcal{R}$  locally reverses the action of  $\text{Tr}_E$  on  $\rho^{QE}$ :  $(\mathcal{R} \circ \text{Tr}_E)(\rho^{QE}) = \rho^{QE}$ , where  $\circ$  denotes the composition of maps.

Henceforth  $\mathcal{R}$  will be referred to as the Petz map. As is apparent from Eq. (4),  $\mathcal{R}$  can be considered to be a composition of two  $CP$  maps: The first is of the form  $B(\mathcal{H}_Q) \rightarrow B(\mathcal{H}_{QE})$  and is defined by  $X \rightarrow (ad_{(\rho^Q)^{-1/2}}X) \otimes \mathbb{I}_E$  and the second is of the form  $B(\mathcal{H}_{QE}) \rightarrow B(\mathcal{H}_{QE})$  and is defined by  $Y \rightarrow ad_{(\rho^{QE})^{1/2}}(Y)$  for any  $Y \in B(\mathcal{H}_{QE})$ . By virtue of the general relation  $\text{Tr}_E[Y(X \otimes \mathbb{I}_E)] = (\text{Tr}_E Y)X$  one can easily verify that

$$\begin{aligned} \text{Tr}_{QE}[\mathcal{R}(X)] &= \text{Tr}_Q \left\{ (\text{Tr}_E \rho^{QE}) [(\rho^Q)^{-1/2} X (\rho^Q)^{-1/2}] \right\} \\ &= \text{Tr}_Q X. \end{aligned}$$

Thus, the Petz map  $\mathcal{R}$  is indeed a quantum channel on the support of  $\rho^Q$ . As it depends on the initial  $\rho^Q$  state,  $\mathcal{R}$  should be indexed by  $\rho^Q$ , but for the sake of clarity this dependence is suppressed. When extended to all of  $B(\mathcal{H}_Q)$  the Petz map can be considered as a trace-non-increasing  $CP$  map [14].

### A. Pure Markov states

For any tripartite pure state  $S(RQE)$  vanishes and according to Schmidt decomposition the bipartite splits  $R|QE, RQ|E$  and  $Q|RE$  imply the following equalities;

$$S(R) = S(QE), \quad S(E) = S(RQ), \quad S(Q) = S(RE).$$

Substituting these relations into Eq. (3) immediately proves the following statement.

*Lemma 1.* For any tripartite pure state the equality  $S(R;E|Q) = S(R;E)$  holds.  $\square$

That is, the conditional mutual information of any tripartite pure state, given  $Q$ , is just the mutual information of  $R$  and  $E$ . Since being a product state is the necessary and sufficient conditions for quantum mutual information of a given bipartite state to vanish, as a corollary of *Lemma 1* we have the following fact which is a general trait of all pure Markov states.

*Corollary 1.* Any tripartite pure state  $\rho^{RQE} = |\psi^{RQE}\rangle\langle\psi^{RQE}|$  is a Markov state iff the marginal state  $\rho^{RE} = \text{Tr}_Q \rho^{RQE}$  is the product state  $\rho^{RE} = \rho^R \otimes \rho^E$ .  $\square$

### B. Canonical form of the pure Markov states

By *Corollary 1*, all tripartite pure Markov states of  $RQE$  are purifications of product states of  $R$  and  $E$ . To say more, let us consider the product state  $\rho^{RE} = \rho^R \otimes \rho^E$ . Denoting the spectra of  $\rho^R$  and  $\rho^E$  by  $\{\kappa_j\}$  and  $\{\mu_k\}$

$$\rho^R = \sum_j \kappa_j |r_j\rangle\langle r_j|, \quad \rho^E = \sum_k \mu_k |\varepsilon_k\rangle\langle \varepsilon_k|, \quad (5)$$

with the corresponding orthonormal eigenstates  $\{|r_j\rangle\}$  and  $\{|\varepsilon_k\rangle\}$  we have [15]

$$|\psi^{RQE}\rangle = \sum_{j,k} \sqrt{\kappa_j \mu_k} |r_j\rangle \otimes |q_{jk}\rangle \otimes |\varepsilon_k\rangle, \quad (6)$$

for purification of  $\rho^R \otimes \rho^E$ . Here  $\{|q_{jk}\rangle; \langle q_{mn}|q_{jk}\rangle = \delta_{jm}\delta_{kn}\}$  represents the set of orthonormal eigenstates of the corresponding state of  $Q$ :

$$\rho^Q = \sum_{j,k} \kappa_j \mu_k |q_{jk}\rangle\langle q_{jk}|. \quad (7)$$

In fact, any tripartite pure state  $|\Phi\rangle$  whose marginal  $\rho^{RE} = \text{Tr}_Q |\Phi\rangle\langle\Phi|$  is a product state is, by definition, a purification of  $\rho^{RE}$ . Moreover  $|\Phi\rangle$  is unique up to local unitary (or, more generally, local isomorphism  $V : \mathcal{H}^Q \rightarrow \mathcal{H}^{Q'}, V^\dagger V = \mathbb{I}_Q$ ) transformations of  $Q$ , that is

$$\rho^{RE} = \text{Tr}_Q |\Phi\rangle\langle\Phi| = \text{Tr}_Q [(id_R \otimes ad_V \otimes id_E) |\Phi\rangle\langle\Phi|].$$

Thus Eq. (6) is a canonical form characterizing all pure Markov states and from the explicit form of diagonal marginal states, or directly from (6) we have the following statement.

*Lemma 2.* For a given pure Markov state the ranks of its one-partite marginal states satisfies the equality:

$$\text{rank}(\rho^Q) = \text{rank}(\rho^E)\text{rank}(\rho^R). \quad \square \quad (8)$$

Some of the immediate corollaries of this Lemma can be directly stated as follows. When both  $R$  and  $E$  are in pure states then so is  $Q$  and we have a pure Markov state as a pure product state. When only one of  $R$  and  $E$  is in a pure state then the pure Markov state has, irrespective of  $\text{rank}(\rho^Q)$ , one of the following form;

$$\rho^{RQE} = \begin{cases} |\varphi^{RQ}\rangle\langle\varphi^{RQ}| \otimes |\psi^E\rangle\langle\psi^E|, \\ |\varphi^R\rangle\langle\varphi^R| \otimes |\psi^{QE}\rangle\langle\psi^{QE}|. \end{cases} \quad (9)$$

When the  $\text{rank}(\rho^Q)$  is a prime number then the above two forms exhaust all possible forms of pure Markov states. In particular, Eqs. (9) exhibit all possible pure Markov states for qubit and qutrit states of  $Q$ . Moreover, for any tripartite pure state of  $RQE$ ,  $\rho^{QE}$  is a pure state iff so is  $\rho^R$ . When  $\rho^R$  is pure the tripartite pure state is automatically a Markov state. Hence the second relation of (9) exhausts the set of pure Markov states in which  $R$  is in a pure state.

In the most general case neither  $QE$  nor  $RQ$  needs to be in a pure state. Indeed, in terms of orthonormal states

$$|\psi_j^{QE}\rangle = \sum_k \sqrt{\mu_k} |q_{jk}\rangle \otimes |\varepsilon_k\rangle, \quad \langle\psi_m^{QE}|\psi_j^{QE}\rangle = \delta_{mj}, \quad (10)$$

$$|\psi_k^{RQ}\rangle = \sum_j \sqrt{\kappa_j} |r_j\rangle \otimes |q_{jk}\rangle, \quad \langle\psi_n^{RQ}|\psi_k^{RQ}\rangle = \delta_{nk},$$

from Eq. (6) we obtain

$$\rho^{QE} = \sum_j \kappa_j |\psi_j^{QE}\rangle\langle\psi_j^{QE}|, \quad \rho^{RQ} = \sum_k \mu_k |\psi_k^{RQ}\rangle\langle\psi_k^{RQ}|.$$

Thus when neither  $R$  nor  $E$  is in pure state, both  $\rho^{QE}$  and  $\rho^{RQ}$  are mixed.

#### IV. PURE MARKOV STATES AND CP EVOLUTIONS

From now on we consider our pure tripartite input state  $\rho^{RQE}$  to be a pure Markov state obeying *Corollary 1*;  $\rho^{RE} = \rho^R \otimes \rho^E$ . To determine the output state we first compute  $\rho^{RQ}$  and write  $\rho^{RQE} = (id_R \otimes \mathcal{R})\rho^{RQ}$ . Then the tripartite output can be written as

$$\begin{aligned} \sigma^{RQE} &= (id_R \otimes ad_{U^{QE}})\rho^{RQE} \\ &= [id_R \otimes (ad_{U^{QE}} \circ \mathcal{R})]\rho^{RQ}. \end{aligned} \quad (11)$$

By writing  $\rho^{RQ}$  in the block form  $\rho^{RQ} = \sum_{ij} e_{ij} \otimes Q_{ij}$ , where  $e_{ij} = |e_i\rangle\langle e_j|$  are the standard matrix units corresponding to the standard unit vectors  $|e_i\rangle$  of  $\mathcal{H}_R$  and

$\rho^Q = \sum_j Q_{jj}$ , Eq. (11) can be rewritten as follows  $\sigma^{RQE} = \sum_{ij} e_{ij} \otimes [U^{QE}\mathcal{R}(Q_{ij})U^{QE\dagger}]$ : Taking partial trace over  $R$  leads us to

$$\sigma^{QE} = (ad_{U^{QE}} \circ \mathcal{R})(\rho^Q). \quad (12)$$

Tracing out the environment and comparing the result with Eq. (1) proves the following statement which is the first main point of our study.

*Theorem 1.* If the initial system-environment state  $\rho^{QE}$  is a marginal (reduced) state of a pure Markov state then the evolution of  $Q$  is described by

$$\sigma^Q = \mathcal{E}(\rho^Q), \quad \mathcal{E} = Tr_E \circ ad_{U^{QE}} \circ \mathcal{R}, \quad (13)$$

where  $\mathcal{R}$  is the Petz map given by Eq. (4).  $\mathcal{E}$  is a CPTP map on the support of initial system state  $\rho^Q$  for every unitary joint evolutions  $U^{QE}$ .  $\square$

Partial traces, adjoint actions by unitary operators (also by isometry operators) and the Petz map are all CPTP maps, that is quantum channels. Since concatenation of quantum channels is again a quantum channel, the evolution map  $\mathcal{E}$  is a quantum channel on the support of  $\rho^Q$ . We should emphasize that in the proof of *Theorem 1* presented in this section no form of a pure Markov state is used, that is, it is also valid for any Markov state and with this general form it can be seen from Theorem 1 of Ref. [9].

#### A. Kraus Operators

In view of Eq. (4) the action  $\mathcal{E}(X) = Tr_E [ad_{U^{QE}} \circ \mathcal{R}(X)]$  of  $\mathcal{E}$  on any  $X \in B(\mathcal{H}_Q)$  can be written as

$$\mathcal{E}(X) = Tr_E \left\{ ad_{U^{QE}(\rho^{QE})^{1/2}} [(ad_{(\rho^Q)^{-1/2}} X) \otimes \mathbb{I}_E] \right\}.$$

Now we consider the orthonormal basis  $\{|\varepsilon_\ell\rangle; \langle\varepsilon_k|\varepsilon_\ell\rangle = \delta_{k\ell}\}$  completing the set of initial eigenvectors of the environment to a complete set. Then by writing  $\sum_\ell |\varepsilon_\ell\rangle\langle\varepsilon_\ell| = \mathbb{I}_E$  and evaluating  $Tr_E$  via the same basis we get the Kraus representation [5]

$$\mathcal{E}(X) = \sum_{k,\ell} ad_{E_{k\ell}}(X), \quad (14)$$

where the Kraus operators are obtained as follows

$$E_{k\ell} = \langle\varepsilon_k|U^{QE}(\rho^{QE})^{1/2}|\varepsilon_\ell\rangle(\rho^Q)^{-1/2}. \quad (15)$$

It should be emphasized that the first index of  $E_{k\ell}$  ranges over the whole basis of  $E$  and second index takes values in the set of initial eigenvectors of  $E$ .

The Kraus operators encapsulate the knowledge of the initial system-environment state and of the joint unitary evolution. It is well-established general fact that a linear map is CP iff it is a sum of adjoint actions generated by a set of Kraus operators. Now, we shall evaluate

the Kraus operators given by Eq. (15) for pure Markov states. Making use of

$$\begin{aligned}(\rho^{QE})^{1/2} &= \sum_j \sqrt{\kappa_j} |\psi_j^{QE}\rangle \langle \psi_j^{QE}|, \\ (\rho^Q)^{-1/2} &= \sum_{i,k} (\kappa_i \mu_k)^{-1/2} |q_{ik}\rangle \langle q_{ik}|,\end{aligned}$$

and  $|\psi_j^{QE}\rangle = \sum_k \sqrt{\mu_k} |q_{jk}\rangle \otimes |\varepsilon_k\rangle$  the Kraus operator can be rewritten as

$$E_{k\ell} = \sum_j \langle \varepsilon_k | U^{QE} |\psi_j^{QE}\rangle \langle q_{j\ell}|. \quad (16)$$

When Eq. (16) is inserted into Eq. (14) we obtain

$$\mathcal{E}(X) = \sum_{i,j} Tr_Q(X \Pi_{ij}) Tr_E(U^{QE} P_{ij}^{QE} U^{QE\dagger}), \quad (17)$$

where we have defined

$$\Pi_{ij} = \sum_\ell |q_{i\ell}\rangle \langle q_{j\ell}|, \quad P_{ij}^{QE} = |\psi_j^{QE}\rangle \langle \psi_i^{QE}|. \quad (18)$$

Noting that  $Tr_{QE} P_{ij}^{QE} = \delta_{ij}$  from Eq. (17) we get  $Tr_Q \mathcal{E}(X) = \sum_i Tr_Q(X \Pi_i)$ , with  $\Pi_i = \Pi_{ii}$ , which implies that  $\mathcal{E}$  is trace-preserving for the operator defined on the support of  $\rho^Q$  denoted by  $supp(Q)$ . Thus, the trace preserving condition is equivalent to  $\sum_{k,\ell} E_{k\ell}^\dagger E_{k\ell} = \mathbb{I}_{supp(Q)}$ . In what follows when  $\mathcal{E}$  is referred to as a channel this support restriction must be understood.

## B. Identification of some channels

In order to identify some special forms of the channel, in terms of the traceless ( $Tr_Q T(X) = 0$ ) linear map  $T$

$$T(X) = \sum_{i \neq j} p_{ij}(X) Tr_E(U^{QE} P_{ij}^{QE} U^{QE\dagger}), \quad (19)$$

and the so called Holevo map

$$\mathcal{E}^H(X) = \sum_i p_i(X) Tr_E(U^{QE} P_i^{QE} U^{QE\dagger}), \quad (20)$$

where  $p_{ij}(X) = Tr_Q(X \Pi_{ij})$ ,  $p_i(X) = Tr_Q(X \Pi_i)$  and  $P_i^{QE} = |\psi_i^{QE}\rangle \langle \psi_i^{QE}|$ , Eq. (17) can be rewritten, more concisely as

$$\mathcal{E}(X) = \mathcal{E}^H(X) + T(X). \quad (21)$$

Eq. (17), or equivalently Eq. (21) provides the most general form of the channel implied by the pure Markov states. These are given in the bases of initial states of  $Q$  and  $E$ . In these bases all operators taking part in both  $\mathcal{E}^H(X)$  and  $T(X)$  are independent from  $X$ . While each operator appearing in the former is a state, entirely determined by the joint unitary evolution and initial pure

$P_i^{QE}$  states, all of operators appearing in the latter are traceless. These forms have also some remarkable special cases to be mentioned.

In particular; when  $p_{ij}(X) = Tr_Q(X \Pi_{ij}) = 0$  for all  $i \neq j$ , or when  $T(X) = 0$  the channel has the form  $\mathcal{E} = \mathcal{E}^H$ . For this reason we would like to call the channel of the form (20) the Holevo channel [16] which is known also as the entanglement breaking channel [17]. In that case the output states are convex mixture of  $\sigma_i^Q = Tr_E(U^{QE} P_i^{QE} U^{QE\dagger})$  such that mixture fractions depends on the input state.

Let us now consider the examples implied by the pure Markov states given by Eq. (9). The first relation  $\rho^{RQE} = |\varphi^{RQ}\rangle \langle \varphi^{RQ}| \otimes |\psi^E\rangle \langle \psi^E|$  of Eq. (9) represents the purification of initial uncorrelated system-environment state considered in the majority of the related literature which provides CPTP evolution for all joint unitary evolutions. This is a pure Markov state in which the environment is in the pure state  $|\psi^E\rangle$ .

On the other hand, in the product state  $|\psi^{RQE}\rangle = |\varphi^R\rangle \otimes |\Psi^{QE}\rangle$  of Eq. (9) we can consider  $|\Psi^{QE}\rangle$  to be a maximally entangled initial state of  $QE$ . Obviously the marginal  $\rho^{RE} = |\varphi^R\rangle \langle \varphi^R| \otimes \mathbb{I}_E/d_E$ , is a product state and  $\rho^Q$  and  $\rho^E$  are maximally mixed. Hence the associated CPTP map is defined on the whole  $B(\mathcal{H}_Q)$ . Since in this case  $R$  is in a pure state, the indices  $i$  and  $j$  take only one value and therefore can be suppressed to obtain

$$\mathcal{E}(X) = Tr_Q(X) Tr_E(U^{QE} P^{QE} U^{QE\dagger}),$$

where  $P^{QE} = |\psi^{QE}\rangle \langle \psi^{QE}|$ . Thus  $\mathcal{E}$  is a completely depolarizing channel mapping all states to a fixed state  $\sigma^Q = Tr_E(U^{QE} P^{QE} U^{QE\dagger})$ . Note that this is a special case of the Holevo channel.

The rest of this study is devoted to characterization of the initial system-environment correlations. It turns out that the entropy  $S(E)$  of the environment play a vital role in this context such that whenever it is nonzero initial states of  $QE$  are correlated.

## V. INITIAL SYSTEM-ENVIRONMENT CORRELATIONS

Our main goal in this section is to specify both quantitatively and qualitatively quantum correlations and classical correlations of the joint system  $QE$  when it is in a marginal state of a pure Markov state. The emphasis will be put on the entanglement of formation, discord and classical correlations and we firstly recall their definitions for a generic bipartite state  $\rho^{AB}$ . Then, relationships between these information theoretical quantities will be established for general tripartite pure states and finally the relationships for the pure Markov states will be deduced from them.

### A. Entanglement of formation (EOF)

When  $\rho^{AB}$  is a pure state we have  $S(AB) = 0$  and  $S(A) = S(B)$ , hence the entropy of a marginal state is a natural quantitative measure of the entanglement of a bipartite pure state. A given bipartite pure state is entangled iff the entropy of its marginal states are different from zero and it is maximally entangled iff the entropy of its marginal states are maximum [3, 6]. For a given mixed state  $\rho^{AB}$  there is not so easy way of even deciding the existence of entanglement. Perhaps the most efficient way is to define, again by means of the entropy of a marginal state, say  $\rho^A$ , the EOF of  $\rho^{AB}$  by [18, 19]

$$E_f(AB) = \inf_{\{p_i, P_i^{AB}\}} \sum_i p_i S(\text{Tr}_B P_i^{AB}), \quad (22)$$

where positive numbers  $p_i$  denotes the probabilities  $\sum_i p_i = 1$  and  $P_i^{AB} = |\psi_i^{AB}\rangle\langle\psi_i^{AB}|$  are rank-1 projectors (pure states) such that  $\langle\psi_i^{AB}|\psi_i^{AB}\rangle = 1$  for all  $i$ , but they do not need to be orthogonal. The *infimum* in Eq. (22) ranges over all possible pure-state decompositions of  $\rho^{AB} = \sum_i p_i P_i^{AB}$ .

### B. An equivalent form of the EOF

For our purpose in this study we should convert the conventional definition of EOF to another equivalent form such that it will be possible to deal with EOF and other measures of correlations on equal footing. For this purpose we shall firstly replace the range of infimum of Eq. (22) with an equivalent set and secondly the average entropy of Eq. (22) will be replaced with a more suitable conditional entropy. To accomplish the first replacement, we note that any pure-state decomposition of a given density matrix can be obtained as the non-selective rank-1 POVM measurements locally carried out on its purifying system. A POVM  $\mathcal{M}$  is a collection of a complete set of the POVM elements  $M_i$  each of which is positive operator and is associated with a single measurement.

Let  $P^{RAB} = |\Phi^{RAB}\rangle\langle\Phi^{RAB}|$  be a purification of a given  $\rho^{AB}$  and let  $\rho^{AB} = \sum_i p_i P_i^{AB}$  be a pure state decomposition of it. We shall denote the rank-1 POVM that produces this decomposition via its nonselective action on  $R$  by the set

$$\mathcal{M} = \{0 \leq M_i \leq \mathbb{I}_R; \sum_i M_i = \mathbb{I}_R\}.$$

Rank-1 POVM means that  $M_i = |\alpha_i\rangle\langle\alpha_i|$  for all  $i$  such that  $|\alpha_i\rangle$ 's need to be neither normalized nor orthogonal. The explicit form of the correspondence between the pure state decomposition and complete local execution of the POVM on the purifying reference system  $R$

can be written as

$$\begin{aligned} \rho^{AB} &= \sum_i p_i P_i^{AB} \\ &= \sum_i \text{Tr}_R[(\sqrt{M_i} \otimes \mathbb{I}_{AB}) P^{RAB} (\sqrt{M_i} \otimes \mathbb{I}_{AB})], \end{aligned}$$

where  $\rho^R = \text{Tr}_{AB} P^{RAB}$  and  $p_i = \text{Tr}_R(M_i \rho^R)$ . Rank-1 condition for all the POVM elements  $M_i$ 's is sufficient (as well as necessary) for the purity of all conditional states  $P_i^{AB}$  (see Appendix A where this statement and its converse are proved).

For the second replacement in the definition of EOF, we observe that in terms of an orthonormal basis  $\{|i\rangle\}$  of an auxiliary Hilbert space  $\mathcal{H}_X$  to any pure-state decompositions (in fact, for any convex mixture) of  $\rho^{AB}$  is associated a classical-quantum state  $\rho^{XAB} = \sum_i p_i |i\rangle\langle i| \otimes P_i^{AB}$ . In such a case, since  $\rho_i^A = \text{Tr}_B P_i^{AB}$  the average entropy  $\sum_i p_i S(\rho_i^A)$  appearing in the definition (22) is nothing more than the conditional entropy  $S(A|X)$  of the state  $\rho^{XA} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i^A$ . Thus we can rewrite EOF given by (22) as follows

$$E_f(AB) = \inf_{\mathcal{M}} \{S(A|X); \rho^{XA} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i^A\}, \quad (23)$$

where  $p_i \rho_i^A = \text{Tr}_{RB}[(M_i \otimes \mathbb{I}_{AB}) P^{RAB}]$ . This is our promise for the new equivalent definition of EOF and it should be emphasized that here the *infimum* must be taken over all rank-1 POVMs acting on  $R$ .

### C. Discord and classical correlations

The classical correlation  $C(R \rightarrow B)$  between  $R$  and  $B$  in the bipartite state  $\rho^{RB} = \text{Tr}_A P^{RAB}$  can be specified by means of the so called Holevo quantity

$$\chi(\{p_i, \rho_i^B\}) = S(\sum_i p_i \rho_i^B) - \sum_i p_i S(\rho_i^B),$$

of the state  $\rho^B = \text{Tr}_{RA} P^{RAB} = \sum_i p_i \rho_i^B$ . The Holevo quantity is a fundamental upper bound for the accessible information between the sender and a receiver communicating classical messages by quantum means (Theorem 5.9 of the Ref. [6]). Here  $\rho_i^B$  is the carrier of the classical message  $i$  send with probability  $p_i$  and the receiver tries to read the message by POVM measurements. A fundamental relation between the classical correlations and EOF can be stated as follows.

*Lemma 3.* For any purification  $\rho^{RAB}$  of a given bipartite state  $\rho^{AB}$ , the EOF  $E_f(AB)$  for  $\rho^{AB}$  and the classical correlations  $C(R \rightarrow B)$  between  $R$  and  $B$  always obey the following equality

$$C(R \rightarrow B) + E_f(AB) = S(B). \quad (24)$$

This relation is well known in the literature as the Koashi-Winter *monogamy* relation [20] for which an alternative and instructive proof can be given as follows.

*Proof.* In terms of the associated classical-quantum state  $\rho^{XB} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i^B$ ,  $\chi(\{p_i, \rho_i^B\})$  can be written as

$$\chi(\{p_i, \rho_i^B\}) = S(X; B) = S(B) - S(B|X).$$

Now, the classical correlations  $C(R \rightarrow B)$  are defined by maximizing the Holevo quantity over all rank-1 POVMs executed on  $R$ ;

$$C(R \rightarrow B) = S(B) - \inf_{\mathcal{M}} S(B|X). \quad (25)$$

Since  $E_f(AB) = E_f(BA)$  the second term at the right hand side is the EOF given by Eq. (23). This, proves Eq. (24).  $\square$

For later convenience in adopting these relations for the pure Markov states, we simply change the label of systems  $A$  and  $B$ , respectively by  $Q$  and  $E$  and consider an arbitrary pure state  $\rho^{RQE}$  which need not be a Markov state yet. For this case we rewrite Eq. (24) as

$$C(R \rightarrow E) + E_f(QE) = S(E). \quad (26)$$

Making use of Eq. (2) and the relations implied by the bipartite splits of  $\rho^{RQE}$  we have

$$\begin{aligned} S(R) &= \frac{1}{2}[S(R; Q) + S(R; E)], \\ S(Q) &= \frac{1}{2}[S(R; Q) + S(Q; E)], \\ S(E) &= \frac{1}{2}[S(R; E) + S(Q; E)]. \end{aligned} \quad (27)$$

These simply state that entropy of any individual part of any tripartite pure state is the arithmetic mean of mutual information of that part with remaining other two partners. Recalling the definition of discord [21]

$$D(Q \rightarrow E) = S(Q; E) - C(Q \rightarrow E) \quad (28)$$

between  $Q$  and  $E$  we can write two important corollaries of Lemma 3.

*Corollary 2.* For any purification  $\rho^{RQE}$  of a given bipartite state  $\rho^{QE}$ , the following equalities hold:

$$C(Q \rightarrow E) + E_f(RE) = S(E), \quad (29)$$

$$S(R; E) - E_f(RE) + D(Q \rightarrow E) = S(E). \quad (30)$$

*Proof.* The first relation is obtained from Eq. (26) by the interchange  $R \leftrightarrow Q$  and the second relation is the difference of third relation of Eq. (27) and Eq. (29).  $\square$

#### D. EOF, discord and classical correlations implied by the pure Markov states

Having specified the quantum mechanical as well as the classical correlations of any pure tripartite state in terms of information theoretical quantities we are ready to apply these results to any pure Markov state  $\rho^{RQE}$ .

Since  $\rho^{RE}$  is a product state in this case the mutual information, the classical correlations and the EOF between  $R$  and  $E$  are zero:

$$S(R; E) = 0 = E_f(RE) = C(R \rightarrow E).$$

Thus Eqs. given by (26), (29) and (30) reduce to the following forms

$$E_f(QE) = C(Q \rightarrow E) = D(Q \rightarrow E) = S(E).$$

That is, numerical values of considered correlations between  $Q$  and  $E$  are equal to each other and their value is simply given by the entropy of the environment. Moreover Eqs. (27) take the forms

$$S(R) = \frac{1}{2}S(R; Q), \quad S(E) = \frac{1}{2}S(Q; E),$$

and  $S(Q) = [S(R; Q) + S(Q; E)]/2$  for any pure Markov state. These observations can be summarized by the following theorem which emphasizes another general trait of the pure Markov states.

*Theorem 2.* For any pure Markov state  $\rho^{RQE}$ , conditional on  $Q$ ; the EOF, discord and classical correlations between the system  $Q$  and its environment  $E$  are all numerically equal to each other and to the entropy of the environment  $E$  which is one-half of the mutual information of  $Q$  and  $E$ .  $\square$

Eqs. (26) and (29) represent a kind of conservation law for the correlations of an environment with the related OQS and purifying system. For a given  $S(E)$  both  $C(R \rightarrow E)$  and  $E_f(QE)$  (or  $C(Q \rightarrow E)$  and  $E_f(RQ)$ ) can change (without exceeding the value  $S(E)$ ) but their sum is always fixed by  $S(E)$ . In the case of a pure Markov state both  $C(Q \rightarrow E)$  and  $E_f(QE)$  are equal to  $S(E)$  which means that as long as  $S(E) \neq 0$ ,  $Q$  and  $E$  are certainly correlated and even in such a case the reduced dynamics of  $Q$  can be described by a CPTP map.

## VI. CONCLUDING REMARKS

Finally we would like to present our remarks concerning two seemingly different topics; **(i)** perfect quantum error correction (QEC) in quantum information theory [22] and **(ii)** quantum non-Markovian (or Markovian) dynamics in the theory of OQSs [1, 23].

**(i)** Main point of the perfect QEC procedure is to faithfully restore the output state of a system  $Q$  to its input state by means of local transformations executed at the receiver side after  $Q$  has been sent through a noisy transmission channel. For the transfer of quantum information the input is supposed to be a part of an entangled pure state with a reference system  $R$  and the noisy channel is modelled to arise, as usually, from a joint unitary evolution of  $Q$  and its environment  $E$  which is supposed to be in a pure state [24]. During the transmission process,  $Q$  evolves as an open system and  $R$  remains intact throughout the process. Thus tripartite framework and

uncorrelated initial  $QE$  state are essential in the present perfect QEC scheme. Moreover, the initial tripartite state is a special pure Markov state corresponding to our first relation of Eq. (9). What is more, being another pure Markov state for the tripartite output suffices for the accomplishment of perfect QEC [22]. Hence perfect QEC procedure can be formulated as preserving the pure Markov structure of the tripartite input state and we do naturally expect that the results of this study can provide a broader perspective on the perfect QEC and on the approximate QEC procedures.

(ii) Quantum non-Markovianity (and Markovianity) is another central topic in the theory of OQSs. This can be defined via divisibility property of evolution maps [23]. A quantum system  $Q$  subject to a time evolution given by some family of trace-preserving linear maps  $\{\mathcal{E}_{\tau',\tau}; \tau' \geq \tau \geq \tau_0\}$  is Markovian (or divisible) iff, for every  $\tau'$  and  $\tau$ ,  $\mathcal{E}_{\tau',\tau}$  is a CPTP map and fulfils the composition law

$$\mathcal{E}_{\tau'',\tau} = \mathcal{E}_{\tau'',\tau'} \circ \mathcal{E}_{\tau',\tau}, \quad \tau'' \geq \tau' \geq \tau. \quad (31)$$

To establish a concrete connection with our work we restore time labelling of initial and final states respectively by  $\tau$  and  $\tau'$  as mentioned in Section II. As we have shown, even in the presence of initial correlations the existence of CPTP evolution  $\mathcal{E}_{\tau',\tau}$  is guaranteed as long as the tripartite initial state is a pure Markov state. Although the tripartite output at time  $\tau'$  is a pure state  $\sigma_{\tau'}^{RQE} = |\phi_{\tau'}^{RQE}\rangle\langle\phi_{\tau'}^{RQE}|$  where  $|\phi_{\tau'}^{RQE}\rangle = (\mathbb{I}_R \otimes U_{\tau',\tau}^{QE})|\psi_{\tau}^{RQE}\rangle$ , it need not be a pure Markov state. Hence there may not be a CP map  $\mathcal{E}_{\tau'',\tau'}$  after the time  $\tau'$ . According to the *Corollary 1*, a sufficient conditions for this to be case is that the reduced output

$$\sigma_{\tau'}^{RE} = Tr_Q[id_R \otimes (ad_{U_{\tau',\tau}^{QE}} \circ \mathcal{R})]\rho_{\tau}^{RQ} \quad (32)$$

being a product state. Whenever this occurs the existence of CPTP map  $\mathcal{E}_{\tau'',\tau'}$  will be guaranteed. Thus, we can say that an evolution which preserves the pure Markov structure, that is, transforming a pure Markov state to another one for all  $\tau'$  is a Markovian evolution for the observed system.

To sum up, the problems such as the reduced quantum dynamics in the presence of initial system-environment correlations, the Markovian and non-Markovian dynamical evolutions and the perfect QEC and approximate QEC procedures are very intimately connected. Pure Markov states, or in a broader context Markov states play a central role in these seemingly different topics.

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### Appendix A

Let  $|\Upsilon^{AR}\rangle$  be a purification of a given state  $\rho^A$  such that  $\rho^A = Tr_R P^{AR}$  with  $P^{AR} = |\Upsilon^{AR}\rangle\langle\Upsilon^{AR}|$  and let  $\mathcal{M} = \{0 \leq M_i \leq \mathbb{I}_R; \sum_i M_i = \mathbb{I}_R\}$ , be a POVM on the purifying system  $R$ . The conditional state  $\rho_i^A$ , known also as the post-measurement state, resulting from a single local measurement is given by

$$p_i \rho_i^A = Tr_R[(\mathbb{I}_A \otimes \sqrt{M_i})P^{AR}(\mathbb{I}_A \otimes \sqrt{M_i})], \quad (A1)$$

where  $p_i = Tr_R(M_i \rho^R)$  is nonzero and  $\rho^R = Tr_A P^{AR}$ . *Lemma A.* The conditional state  $\rho_i^A$  defined by (A1) is a pure state iff the POVM element  $M_i$  is of rank-1 [25].

*Proof.* Let  $\{|\psi_m\rangle\}$  be the set of the orthonormal eigenvectors of  $\rho^A$  corresponding to nonzero eigenvalues  $\{\lambda_m\}$  and let  $\{|m\rangle\}$  be an orthonormal basis of  $R$ . By writing

$$P^{AR} = \sum_{mn} \sqrt{\lambda_m \lambda_n} |\psi_m\rangle\langle\psi_n| \otimes |m\rangle\langle n|,$$

we have, from (A1)

$$p_i \rho_i^A = \sum_{mn} \sqrt{\lambda_m \lambda_n} |\psi_m\rangle\langle\psi_n| Tr_R(M_i |m\rangle\langle n|). \quad (A2)$$

When  $M_i$  is of rank-1 we can write  $M_i = |\alpha_i\rangle\langle\alpha_i|$  where nonzero  $|\alpha_i\rangle$  need not be normalized. Then by defining  $c_{im} = \langle\alpha_i|m\rangle$  from (A2) we have  $p_i \rho_i^A = \sum_{mn} c_{im} c_{in}^* \sqrt{\lambda_m \lambda_n} |\psi_m\rangle\langle\psi_n|$ , where  $*$  denotes the complex conjugation, and in terms of  $|\Phi_i\rangle = \sum_m c_{im} \sqrt{\lambda_m} |\psi_m\rangle$  we obtain  $p_i \rho_i^A = |\Phi_i\rangle\langle\Phi_i|$ . Since  $\rho^R = \sum_m \lambda_m |m\rangle\langle m|$  and

$$p_i = Tr_R(M_i \rho^R) = \sum_m \lambda_m |c_{im}|^2 = \langle\Phi_i|\Phi_i\rangle,$$

in terms of the normalized state  $|\varphi_i\rangle = |\Phi_i\rangle/\sqrt{p_i}$  we have  $\rho_i^A = |\varphi_i\rangle\langle\varphi_i|$ . Conversely, suppose that  $\rho_i^A$  is a pure state such that  $\rho_i^A = |\beta_i\rangle\langle\beta_i|$  with  $\langle\beta_i|\beta_i\rangle = 1$ . Then from (A2) we obtain

$$p_i |\beta_i\rangle\langle\beta_i| = \sum_{mn} \sqrt{\lambda_m \lambda_n} |\psi_m\rangle\langle\psi_n| (M_i)_{nm}. \quad (A3)$$

Here  $(M_i)_{nm} = \langle n|M_i|m\rangle$  denotes the matrix element of  $M_i$  in the orthonormal basis  $\{|m\rangle\}$  of  $R$ . In terms of  $b_{ik} = \sqrt{p_i/\lambda_k} \langle\psi_k|\beta_i\rangle$  from (A3) we obtain  $(M_i)_{k\ell} = b_{i\ell}^* b_{ik}$  which implies  $M_i = |b_i\rangle\langle b_i|$ , with  $\langle b_i|m\rangle = b_{im}$ .  $\square$



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